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Iterative approximation for split common fixed point problem involving an asymptotically nonexpansive semigroup and a total asymptotically strict pseudocontraction

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Abstract
In this paper, we prove the strong convergence theorem for split feasibility problem involving a uniformly asymptotically regular nonexpansive semigroup and a total asymptotically strict pseudocontractive mapping in Hilbert spaces. Our main results improve and extend some recent results in the literature.
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1 Introduction

In this paper, we assume that H is a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the norm $\|\cdot\|$. Let I denote the identity operator on H. Let C and Q be nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2, respectively. The split feasibility problem (SFP) is to find a point

$$x \in C \text{ such that } Ax \in Q, \tag{1.1}$$

where $A : H_1 \to H_2$ is a bounded linear operator. The SFP in finite-dimensional Hilbert spaces was first introduced by Censor and Elfving [1] for modeling inverse problems which arise from phase retrievals and in medical image reconstruction [2]. The SFP attracts the attention of many authors due to its application in signal processing. Various algorithms have been invented to solve it (see, for example, [3–11] and references therein).

Note that the split feasibility problem (1.1) can be formulated as a fixed point equation by using the fact

$$P_C\big(I - \gamma A^*(I - P_Q)A\big)x^* = x^*; \tag{1.2}$$

that is, x^* solves SFP (1.1) if and only if x^* solves fixed point equation (1.2) (see [8] for details). This implies that we can use fixed point algorithms (see [12–14]) to solve SFP. A popular algorithm that solves SFP (1.1) is due to Byrne's CQ algorithm [2] which is found to be a gradient-projection method (GPM) in convex minimization. Subsequently, Byrne [3] applied KM iteration to the CQ algorithm, and Zhao and Yang [15] applied KM iteration to the perturbed CQ algorithm to solve the SFP. It is well known that the CQ algorithm and the KM algorithm for a split feasibility problem do not necessarily converge strongly in the infinite-dimensional Hilbert spaces.

Now let us recall the definitions of some operators that will be used in this paper.

Let $T : H \to H$ be a mapping. A point $x \in H$ is said to be a *fixed point* of T provided that $Tx = x$, and denote by $F(T)$ the fixed point set of T.

Definition 1.1 The mapping $T : H \to H$ is said to be

(a) *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H;$$

(b) *strictly pseudocontractive* if there exists a constant $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\big\|(x - y) - (Tx - Ty)\big\|^2, \quad \forall x, y \in H;$$

(c) $(k, \{\mu_n\}, \{\xi_n\}, \phi)$-*total asymptotically strict pseudocontractive* if there exists a constant $k \in [0, 1)$ and sequences $\{\mu_n\} \subset [0, \infty)$, $\{\xi_n\} \subset [0, \infty)$ with $\mu_n \to 0$ and $\xi_n \to 0$ as $n \to \infty$, and a continuous and strictly increasing function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that for all $n \geq 1$, $x, y \in H$,

$$\big\|T^n x - T^n y\big\|^2 \leq \|x - y\|^2 + k\big\|(x - y) - (Tx - Ty)\big\|^2 + \mu_n \phi\big(\|x - y\|\big) + \xi_n, \quad \forall x, y \in H.$$

One parameter family $\Gamma := \{S(t) : 0 \leq t < \infty\}$ is said to be a (*continuous*) *Lipschitzian semigroup* on a real Hilbert space H if the following conditions are satisfied:

(1) $S(0)x = x$ for all $x \in H$;
(2) $S(s + t) = S(s)S(t)$ for all $s, t \geq 0$;
(3) for each $t > 0$, there exists a bounded measurable function $L_t : (0, \infty) \to [0, \infty)$ such that

$$\big\|S(t)x - S(t)y\big\| \leq L_t\|x - y\|, \quad x, y \in H;$$

(4) for each $x \in H$, the mapping $S(\cdot)x$ from $[0, \infty)$ into H is continuous.

A Lipschitzian semigroup Γ is called *nonexpansive* (*or contractive*) if $L_t = 1$ for all $t > 0$ and *asymptotically nonexpansive* if $\limsup_{t\to\infty} L_t \leq 1$, respectively. Let $F(\Gamma)$ denote the common fixed point set of the semigroup Γ, *i.e.*, $F(\Gamma) := \{x \in K : S(t)x = x, \forall t > 0\}$.

Let H be a real Hilbert space, $\Gamma := \{S(t) : 0 \leq t < \infty\}$ be a continuous operator semigroup on H. Then Γ is said to be *uniformly asymptotically regular* (in short, u.a.r.) on H if for all $h \geq 0$ and any bounded subset C of H,

$$\lim_{t\to\infty} \sup_{x\in C}\big\|S(h)\big(S(t)x\big) - S(t)x\big\| = 0.$$

The nonexpansive semigroup $\{\sigma_t : t > 0\}$ defined by the following lemma is an example of u.a.r. operator semigroup. Other examples of u.a.r. operator semigroup can be found in Examples 17, 18 of [16].

Lemma 1.2 (See Lemma 2.7 of [17]) *Let D be a bounded closed convex subset of H, and $\Gamma := \{S(t) : t > 0\}$ be a nonexpansive semigroup on H such that $F(\Gamma)$ is nonempty. For each $h > 0$, set $\sigma_t(x) = \frac{1}{t}\int_0^t S(s)x\,ds$, then*

$$\lim_{t\to\infty}\sup_{x\in D}\bigl\|S(h)(\sigma_t x) - \sigma_t x\bigr\| = 0.$$

Example 1.3 (See [18]) The set $\{\sigma_t : t > 0\}$ defined by Lemma 1.2 is an u.a.r. nonexpansive semigroup.

Several authors have proved several convergence theorems using several iterative schemes for fixed points of nonexpansive semigroups in the literature. See, for example, [16–22] and the references contained therein.

In this paper, we shall focus our attention on the following split common fixed point problem (SCFP):

$$\text{find } x \in C \text{ such that } Ax \in Q, \tag{1.3}$$

where $A : H_1 \to H_2$ is a bounded linear operator, $\{S(t) : t \geq 0\}$ is a uniformly asymptotically regular nonexpansive semigroup on H_1 and T is a uniformly L-Lipschitzian continuous and $(k, \{\mu_n\}, \{\xi_n\}, \phi)$-total asymptotically strict pseudocontractive mapping with nonempty fixed point sets $C := \bigcap_{t\geq 0} F(S(t))$ and $Q := F(T)$, and denote the solution set of the two-operator SCFP by

$$\Omega := \{y \in C : Ay \in Q\} = C \cap A^{-1}(Q). \tag{1.4}$$

Recall that $\bigcap_{t\geq 0} F(S(t))$ and $F(T)$ are closed and convex subsets of H_1 and H_2, respectively. If $\Omega \neq \emptyset$, we have that Ω is a closed and convex subset of H_1. The split common fixed point problem (SCFP) is a generalization of the split feasibility problem (SFP) and the convex feasibility problem (CFP) (see [2, 23]).

In order to solve (1.3), Censor and Segal [23] proposed and proved, in finite-dimensional spaces, the convergence of the following algorithm:

$$x_{n+1} = S\bigl(x_n + \gamma A^t(T - I)Ax_n\bigr), \quad n \geq 1, \tag{1.5}$$

where $\gamma \in (0, \frac{2}{\lambda})$, with λ being the largest eigenvalue of the matrix A^tA (A^t stands for matrix transposition) and S and T are quasi-nonexpansive operators.

In 2011, Moudafi [5] introduced the following relaxed algorithm:

$$x_{n+1} = (1 - \alpha_n)y_n + \alpha_n S y_n, \quad n \geq 1, \tag{1.6}$$

where $y_n = x_n + \gamma A^*(T - I)Ax_n$, $\beta \in (0,1)$, $\alpha_n \in (0,1)$, and $\gamma \in (0, \frac{1}{\lambda\beta})$, with λ being the spectral radius of the operator A^*A. Moudafi proved weak convergence result of algorithm (1.6) in Hilbert spaces where S and T are quasi-nonexpansive operators. We observe that strong convergence result can be obtained in the results of Moudafi [5] if a compactness-type condition like demicompactness is imposed on the operator S. Furthermore, we can also obtain a strong convergence result by suitably modifying algorithm (1.6).

Recently, Zhao and He [24] introduced the following viscosity approximation algorithm

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)\big((1-w_n)x_n + w_n S\big(x_n + \gamma A^*(T-I)Ax_n\big)\big), \quad n \geq 1, \tag{1.7}$$

where $f : H_1 \to H_1$ is a contraction of modulus $\rho > 0$, $w_n \in (0, \frac{1}{2})$, $\gamma \in (0, \frac{1}{\lambda})$, with λ being the spectral radius of the operator A^*A, and they proved strong convergence results concerning (1.3) for quasi-nonexpansive operators S and T in real Hilbert spaces. Inspired by the work of Zhao and He [24], Moudafi [6] quite recently revisited the viscosity-type approximation method (1.7) above introduced in [24]. First, he proposed a simple proof of the strong convergence of the iterative sequence $\{x_n\}$ defined by (1.7) based on attracting operator properties and then proposed a modification of this algorithm (1.7) and proved its strong convergence (see Theorem 2.1 of [6]).

Very recently Chang *et al.* [25] proved the following convergence theorem for *multiple-set split feasibility problem* (*MSSFP*) (1.3) for a family of multi-valued quasi-nonexpansive mappings and a total asymptotically pseudocontractive mapping in infinitely dimensional Hilbert spaces.

Theorem 1.4 *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator and $A^* : H_2 \to H_1$ be the adjoint of A. Let $\{S_i\}_{i=1}^{\infty} : H_1 \to CB(H_1)$ be a family of multi-valued quasi-nonexpansive mappings and for each $i \geq 1$, S_i is demiclosed at* 0. *Let $T : H_2 \to H_2$ be a uniformly L-Lipschitzian continuous and $(k, \{\mu_n\}, \{\xi_n\}, \phi)$-total asymptotically strict pseudocontractive mapping satisfying $\sum_{n=1}^{\infty} \mu_n < \infty$ and $\sum_{n=1}^{\infty} \xi_n < \infty$. Suppose that there exist constants $M_0 > 0$, $M_1 > 0$ such that $\phi(\lambda) \leq M_0\lambda^2$, $\forall \lambda > M_1$. Let $C := \bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$ and $Q := F(T)$. Assume that for each $p \in C$, $S_i p = \{p\}$ for each $i \geq 1$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_1 \in H_1 \quad \text{chosen arbitrarily}, \\ x_{n+1} = \alpha_{0,n} y_n + \sum_{i=1}^{\infty} \alpha_{i,n} w_{i,n}, \quad w_{i,n} \in S_i y_n, \\ y_n = x_n + \gamma A^*(T^n - I)Ax_n, \quad n \geq 1, \end{cases} \tag{1.8}$$

where $\{\alpha_{i,n}\} \subset (0,1)$ and $\gamma > 0$ satisfy the following conditions:

(a) $\sum_{i=0}^{\infty} \alpha_{i,n} = 1$ *for each* $n \geq 1$;
(b) *for each* $i \geq 1$, $\liminf_{n\to\infty} \alpha_{0,n}\alpha_{i,n} > 0$;
(c) $\gamma \in (0, \frac{1-k}{\|A\|^2})$.

If Ω (the set of solutions of multiple-set split feasibility problem (1.3)) *is nonempty, then both $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ converge weakly to some point $x \in \Omega$. In addition, if there exists a positive integer m such that S_m is semi-compact, then both $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ converge strongly to $x \in \Omega$.*

We observe on Theorem 1.4 that:

(1) Theorem 1.4 gives a *weak convergence result* for multiple-set split feasibility problem (1.3) for a family of multi-valued quasi-nonexpansive mappings and a total asymptotically pseudocontractive mapping in infinitely dimensional Hilbert spaces. In order to get strong convergence, Chang *et al.* [25] imposed a compactness-type condition (semi-compactness) on the mappings $\{S_i\}_{i=1}^{\infty}$. This compactness condition *appears strong* as only few mappings are semi-compact.

(2) It is an interesting problem to extend the results of Theorem 1.4 to the nonexpansive semigroup case so that *strong convergence* is obtained. In order to obtain a strong convergence result in Theorem 1.4 without compactness-type condition for the nonexpansive semigroup case, a modification of (1.8) is necessary. This modification could be an implicit iterative scheme or an explicit iterative scheme. In the implicit iterative scheme, the computation of the next iteration x_{n+1} involves solving a nonlinear equation at every step of the iteration, a task which may pose the same difficulty level as the initial problem. Therefore, in order to get a strong convergence result for the split common fixed point problem for a nonexpansive semigroup case and a total asymptotically pseudocontractive mapping in infinitely dimensional Hilbert spaces without compactness-type condition, a modification of (1.8), which is an explicit iterative scheme, is necessary. This leads to the following natural question.

Question Can we modify the iterative scheme (1.8) so that strong convergence is guaranteed for a split common fixed point problem involving a uniformly asymptotically regular nonexpansive semigroup and a total asymptotically pseudocontractive mapping in infinitely dimensional Hilbert spaces without any compactness-type condition assumed?

Our interest in this paper is to answer the above question. We thus modify the iterative scheme (1.8) and prove a strong convergence result for the split common fixed point problem for a uniformly asymptotically regular nonexpansive semigroup and a total asymptotically pseudocontractive mapping in infinitely dimensional Hilbert spaces without any further compactness-type condition assumed. Our results improve the corresponding results of Chang *et al.* [25] and many recent and important results that the results of Chang *et al.* [25] improved and extended like Censor *et al.* [26, 27], Yang [10], Moudafi [28], Xu [9], Censor and Segal [23], Masad and Reich [29] and others.

2 Preliminaries

We first recall some definitions, notations and conclusions which will be needed in proving our main results.

- $x_n \to x$ means that $x_n \to x$ strongly;
- $x_n \rightharpoonup x$ means that $x_n \to x$ weakly.

Next, we state the following well-known lemmas which will be used in the sequel.

Lemma 2.1 *Let H be a real Hilbert space. Then the following well-known results hold*:

(i) $\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$, $\forall x, y \in H$;

(ii) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$, $\forall x, y \in H$;

(iii) $\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|$, $\forall x, y \in H$, $\forall \lambda \in [0, 1]$.

Lemma 2.2 (Chang *et al.* [25]) *Let $T : H \to H$ be a uniformly L-Lipschitzian continuous and $(k, \{\mu_n\}, \{\xi_n\}, \phi)$-total asymptotically strict pseudocontractive mapping, then $I - T$ is demiclosed at origin*.

Lemma 2.3 (Alber *et al.* [30]) *Let $\{\lambda_n\}$ and $\{\gamma_n\}$ be nonnegative, $\{\alpha_n\}$ be positive real numbers such that*

$$\lambda_{n+1} \leq \lambda_n - \alpha_n\lambda_n + \gamma_n, \quad n \geq 1.$$

Let for all $n > 1$,

$$\frac{\gamma_n}{\alpha_n} \leq c_1 \quad \textit{and} \quad \alpha_n \leq \alpha.$$

Then $\lambda_n \leq \max\{\lambda_1, K_*\}$, *where* $K_* = (1+\alpha)c_1$.

Lemma 2.4 (Xu [31]) *Let* $\{a_n\}$ *be a sequence of nonnegative real numbers satisfying the following relation*:

$$a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \quad n \geq 0,$$

where
(i) $\{a_n\} \subset [0,1]$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(ii) $\limsup_{n\to\infty} \sigma_n \leq 0$;
(iii) $\gamma_n \geq 0$, $\sum_{n=1}^{\infty} \gamma_n < \infty$.
Then $a_n \to 0$ *as* $n \to \infty$.

3 Main results

For solving the split common fixed point problem (1.3), we assume that the following conditions are satisfied:

(1) H_1 and H_2 are two real Hilbert spaces, $A : H_1 \to H_2$ is a bounded linear operator and $A^* : H_2 \to H_1$ is the adjoint of A;
(2) $\{S(t) : t \geq 0\}$ is a uniformly asymptotically regular nonexpansive semigroup on H_1;
(3) $T : H_2 \to H_2$ is a uniformly L-Lipschitzian continuous and $(k, \{\mu_n\}, \{\xi_n\}, \phi)$-total asymptotically strict pseudocontractive mapping satisfying the following conditions:
   (i) $\sum_{n=1}^{\infty} \mu_n < \infty$; $\sum_{n=1}^{\infty} \xi_n < \infty$;
   (ii) $\{\alpha_n\}$ is a real sequence in $(0,1)$ such that $\mu_n = o(\alpha_n)$, $\xi_n = o(\alpha_n)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;
   (iii) there exist constants $M_0 > 0$, $M_1 > 0$ such that $\phi(\lambda) \leq M_0\lambda^2$, $\forall \lambda > M_1$;
   (iv) $C := \bigcap_{t\geq 0} F(S(t)) \neq \emptyset$, $Q := F(T) \neq \emptyset$ and $\Omega \neq \emptyset$.

In this section, we introduce the following algorithm and prove its strong convergence for solving split common fixed point problem (1.3).

Theorem 3.1 *Let* H_1, H_2, A, A^*, $\{S(t) : t \geq 0\}$, T, C, Q, k, $\{\mu_n\}$, $\{\xi_n\}$, ϕ *and* L *satisfy the above conditions* (i)-(iv). *Let* $\{x_n\}$ *be the sequence generated by* $x_1 \in H_1$,

$$\begin{cases} u_n = (1-\alpha_n)x_n, \\ y_n = u_n + \gamma A^*(T^n - I)Au_n, \\ x_{n+1} = \beta_n y_n + (1-\beta_n)S(t_n)y_n, \quad n \geq 1, \end{cases} \tag{3.1}$$

where $t_n \to \infty$ *and* $\{\beta_n\} \subset (0,1)$ *and* $\gamma > 0$ *satisfy the following conditions*:
(a) $0 < \epsilon \leq \beta_n \leq b < 1$;
(b) $\gamma \in (0, \frac{1-k}{\|A\|^2})$.
If Ω *is nonempty, then the sequence* $\{x_n\}_{n=1}^{\infty}$ *converges strongly to an element of* Ω.

Proof Since ϕ is continuous, it follows that ϕ attains maximum (say M) in $[0, M_1]$ and by our assumption, $\phi(\lambda) \leq M_0\lambda^2$, $\forall \lambda > M_1$. In either case, we have that

$$\phi(\lambda) \leq M + M_0\lambda^2, \quad \forall \lambda \in [0, \infty).$$

Let $x^* \in \Omega$. Then, by the convexity of $\|\cdot\|^2$, we obtain

$$\begin{aligned} \|u_n - x^*\|^2 &= \|(1-\alpha_n)x_n - x^*\|^2 = \|(1-\alpha_n)(x_n - x^*) + \alpha_n(-x^*)\|^2 \\ &\leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\|x^*\|^2. \end{aligned} \tag{3.2}$$

From (3.1) and Lemma 2.1(i), we obtain that

$$\begin{aligned} \|y_n - x^*\|^2 &= \|u_n - x^* + \gamma A^*(T^n - I)Au_n\|^2 \\ &= \|u_n - x^*\|^2 + 2\gamma\langle u_n - x^*, A^*(T^n - I)Au_n\rangle + \gamma^2\|A^*(T^n - I)Au_n\|^2. \end{aligned} \tag{3.3}$$

Since

$$\begin{aligned} \gamma^2\|A^*(T^n - I)Au_n\|^2 &= \gamma^2\langle A^*(T^n - I)Au_n, A^*(T^n - I)Au_n\rangle \\ &= \gamma^2\langle AA^*(T^n - I)Au_n, (T^n - I)Au_n\rangle \\ &\leq \gamma^2\|A\|^2\|(T^n - I)Au_n\|^2, \end{aligned} \tag{3.4}$$

$Ax^* \in Q = F(T)$ and T is a total asymptotically strict pseudocontractive mapping, then we obtain

$$\begin{aligned} &\langle u_n - x^*, A^*(T^n - I)Au_n\rangle \\ &\quad= \langle A(u_n - x^*), (T^n - I)Au_n\rangle \\ &\quad= \langle A(u_n - x^*) + (T^n - I)Au_n - (T^n - I)Au_n, (T^n - I)Au_n\rangle \\ &\quad= \langle T^nAu_n - Ax^*, (T^n - I)Au_n\rangle - \|(T^n - I)Au_n\|^2 \\ &\quad= \frac{1}{2}\big[\|T^nAu_n - Ax^*\|^2 + \|(T^n - I)Au_n\|^2 - \|Au_n - Ax^*\|^2\big] \\ &\qquad - \|(T^n - I)Au_n\|^2 \\ &\quad\leq \frac{1}{2}\big[\|Au_n - Ax^*\|^2 + k\|(T - I)Ax_n\|^2 + \mu_n\phi(\|Au_n - Ax^*\|) + \xi_n\big] \\ &\qquad + \frac{1}{2}\big[\|(T^n - I)Au_n\|^2 - \|Au_n - Ax^*\|^2\big] \\ &\qquad - \|(T^n - I)Au_n\|^2 \\ &\quad\leq \frac{k-1}{2}\|(T^n - I)Au_n\|^2 + \frac{\mu_n}{2}(M + M_0\|Au_n - Ax^*\|^2) + \frac{\xi_n}{2}. \end{aligned} \tag{3.5}$$

Substituting (3.5) and (3.4) into (3.3), we have

$$\begin{aligned} \|y_n - x^*\|^2 \leq{}& \|u_n - x^*\|^2 - \gamma(1 - k - \gamma\|A\|^2)\|(T^n - I)Au_n\|^2 \\ &+ \mu_n\gamma(M + M_0\|Au_n - Ax^*\|^2) + \gamma\xi_n. \end{aligned} \tag{3.6}$$

Putting (3.6) and (3.2) into (3.1), we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\beta_n(y_n - x^*) + (1-\beta_n)(S(t_n)y_n - x^*)\|^2 \\
&\le \beta_n\|y_n - x^*\|^2 + (1-\beta_n)\|S(t_n)y_n - x^*\|^2 \\
&\le \beta_n\|y_n - x^*\|^2 + (1-\beta_n)\|y_n - x^*\|^2 = \|y_n - x^*\|^2 \\
&\le \|u_n - x^*\|^2 - \gamma(1-k-\gamma\|A\|^2)\|(T^n - I)Au_n\|^2 \\
&\quad + \mu_n\gamma(M + M_0\|Au_n - Ax^*\|^2) + \gamma\xi_n \\
&\le (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\|x^*\|^2 - \gamma(1-k-\gamma\|A\|^2)\|(T^n - I)Au_n\|^2 \\
&\quad + \mu_n\gamma(M + M_0\|A\|^2((1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\|x^*\|^2)) + \gamma\xi_n \\
&= \|x_n - x^*\|^2 - (\alpha_n - \mu_n\gamma M_0\|A\|^2(1-\alpha_n))\|x_n - x^*\|^2 + \alpha_n\|x^*\|^2 \\
&\quad + \mu_n\gamma M + \gamma\alpha_n M_0\|A\|^2\|x^*\|^2 + \gamma\xi_n \\
&\quad - \gamma(1-k-\gamma\|A\|^2)\|(T^n - I)Au_n\|^2 \\
&\le \|x_n - x^*\|^2 - (\alpha_n - \mu_n\gamma M_0\|A\|^2(1-\alpha_n))\|x_n - x^*\|^2 + \alpha_n\|x^*\|^2 \\
&\quad + \mu_n\gamma M + \gamma\alpha_n M_0\|A\|^2\|x^*\|^2 + \gamma\xi_n \\
&= \|x_n - x^*\|^2 - (\alpha_n - \mu_n\gamma M_0\|A\|^2(1-\alpha_n))\|x_n - x^*\|^2 + \sigma_n, \qquad (3.7)
\end{aligned}$$

where $\sigma_n = \alpha_n\|x^*\|^2\mu_n\gamma M + \gamma\alpha_n M_0\|A\|^2\|x^*\|^2 + \gamma\xi_n$. Since $\mu_n = o(\alpha_n)$ and $\xi_n = o(\alpha_n)$, we may assume without loss of generality that there exist constants $k_0 \in (0,1)$ and $M_2 > 0$ such that for all $n \ge 1$,

$$\frac{\mu_n}{\alpha_n} \le \frac{1-k_0}{M_0\gamma\|A\|^2(1-\alpha_n)} \quad \text{and} \quad \frac{\sigma_n}{\alpha_n} \le M_2.$$

Thus, we obtain

$$\|x_{n+1} - x^*\|^2 \le \|x_n - x^*\|^2 - \alpha_n k_0\|x_n - x^*\|^2 + \sigma_n.$$

By Lemma 2.3, we have that

$$\|x_n - x^*\|^2 \le \max\{\|x_1 - x^*\|^2, (1+k_0)M_2\}.$$

Therefore, $\{x_n\}$ is bounded. Furthermore, the sequences $\{y_n\}$ and $\{u_n\}$ are bounded.

The rest of the proof will be divided into two parts.

Case 1. Suppose that there exists $n_0 \in \mathbb{N}$ such that $\{\|x_n - x^*\|\}_{n=n_0}^{\infty}$ is nonincreasing. Then $\{\|x_n - x^*\|\}_{n=1}^{\infty}$ converges and

$$\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \to 0, \quad n \to \infty.$$

From (3.6), we have that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \|y_n - x^*\|^2 \\
&\le \|x_n - x^*\|^2 - \gamma(1-k-\gamma\|A\|^2)\|(T^n - I)Ax_n\|^2
\end{aligned}$$

$$\begin{aligned}
&+ \mu_n\gamma\left(M + M_0\left\|Ax_n - Ax^*\right\|^2\right) + \gamma\xi_n \\
\leq{}& \left\|x_n - x^*\right\|^2 - \gamma\left(1 - k - \gamma\|A\|^2\right)\left\|\left(T^n - I\right)Ax_n\right\|^2 \\
&+ \mu_n\gamma\left(M + M_0\left\|Ax_n - Ax^*\right\|^2\right) + \gamma\xi_n + \alpha_n\left\|x^*\right\|^2.
\end{aligned}$$

This implies that

$$\begin{aligned}
\gamma\left(1 - k - \gamma\|A\|^2\right)\left\|\left(T^n - I\right)Ax_n\right\|^2 \leq{}& \left\|x_n - x^*\right\|^2 - \left\|x_{n+1} - x^*\right\|^2 \\
&+ \mu_n\gamma\left(M + M_0\left\|Ax_n - Ax^*\right\|^2\right) + \gamma\xi_n + \alpha_n\left\|x^*\right\|^2
\end{aligned}$$

and

$$\gamma\left(1 - k - \gamma\|A\|^2\right)\left\|\left(T^n - I\right)Ax_n\right\|^2 \to 0, \quad n \to \infty.$$

Hence, we obtain

$$\left\|\left(T^n - I\right)Ax_n\right\| \to 0, \quad n \to \infty. \tag{3.8}$$

Also, we observe that

$$\|y_n - x_n\| = \gamma A^*\left\|\left(T^n - I\right)Ax_n\right\| \to 0, \quad n \to \infty.$$

Using (3.6) and Lemma 2.1(iii) in (3.1), we have

$$\begin{aligned}
\left\|x_{n+1} - x^*\right\|^2 ={}& \left\|(1 - \beta_n)\left(y_n - x^*\right) + \beta_n\left(S(t_n)y_n - x^*\right)\right\|^2 \\
\leq{}& (1 - \beta_n)\left\|y_n - x^*\right\|^2 + \beta_n\left\|S(t_n)y_n - x^*\right\|^2 - (1 - \beta_n)\beta_n\left\|y_n - S(t_n)y_n\right\| \\
={}& \left\|y_n - x^*\right\|^2 - (1 - \beta_n)\beta_n\left\|y_n - S(t_n)y_n\right\| \\
\leq{}& \left\|x_n - x^*\right\|^2 - \gamma\left(1 - k - \gamma\|A\|^2\right)\left\|\left(T^n - I\right)Ax_n\right\|^2 \\
&+ \mu_n\gamma\left(M + M_0\left\|Ax_n - Ax^*\right\|^2\right) + \gamma\xi_n - (1 - \beta_n)\beta_n\left\|y_n - S(t_n)y_n\right\|.
\end{aligned}$$

This implies from (3.2) and condition (b) that

$$\begin{aligned}
(1 - \beta_n)\beta_n\left\|y_n - S(t_n)y_n\right\| \leq{}& \left\|x_n - x^*\right\|^2 - \left\|x_{n+1} - x^*\right\|^2 \\
&+ \mu_n\gamma\left(M + M_0\left\|Ax_n - Ax^*\right\|^2\right) + \gamma\xi_n \\
\leq{}& \left\|x_n - x^*\right\|^2 - \left\|x_{n+1} - x^*\right\|^2 + \alpha_n\left\|x^*\right\|^2 \\
&+ \mu_n\gamma\left(M + M_0\left\|Ax_n - Ax^*\right\|^2\right) + \gamma\xi_n.
\end{aligned}$$

From condition (a) we have

$$\lim_{n\to\infty}\left\|y_n - S(t_n)y_n\right\| = 0. \tag{3.9}$$

Hence, for any $t \geq 0$,

$$\begin{aligned}
\left\|S(t)y_n - y_n\right\| \leq{}& \left\|S(t)y_n - S(t)S(t_n)y_n\right\| + \left\|S(t)S(t_n)y_n - S(t_n)y_n\right\| + \left\|S(t_n)y_n - y_n\right\| \\
\leq{}& \left\|S(t)S(t_n)y_n - S(t_n)y_n\right\| + 2\left\|S(t_n)y_n - y_n\right\| \to 0, \quad n \to \infty. \qquad (3.10)
\end{aligned}$$

We obtain from (3.1) that

$$\begin{aligned}\|x_{n+1}-x_n\|^2 &= \big\|\beta_n(y_n-x_n)+(1-\beta_n)\big(S(t_n)y_n-x_n\big)\big\|^2\\ &\le \beta_n\|y_n-x_n\|^2+(1-\beta_n)\big\|S(t_n)y_n-x_n\big\|^2\\ &\le \beta_n\|y_n-x_n\|^2+(1-\beta_n)\big(\big\|S(t_n)y_n-y_n\big\|+\|y_n-x_n\|\big)^2.\end{aligned}$$

Since $\lim_{n\to\infty}\|y_n-x_n\|=0$ and $\lim_{n\to\infty}\|y_n-S(t_n)y_n\|=0$, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.$$

Consequently,

$$\begin{aligned}\|u_{n+1}-u_n\| &= \big\|(1-\alpha_{n+1})x_{n+1}-(1-\alpha_n)x_n\big\|\\ &\le |\alpha_{n+1}-\alpha_n|\|x_{n+1}\|+(1-\alpha_n)\|x_{n+1}-x_n\|\to 0,\quad n\to\infty.\end{aligned}\tag{3.11}$$

Using the fact that T is uniformly L-Lipschitzian, we have

$$\begin{aligned}\|TAu_n-Au_n\| &\le \big\|TAu_n-T^{n+1}Au_n\big\|+\big\|T^{n+1}Au_n-T^{n+1}Au_{n+1}\big\|\\ &\quad+\big\|T^{n+1}Au_{n+1}-Au_{n+1}\big\|+\|Au_{n+1}-Au_n\|\\ &\le L\big\|Au_n-T^nAu_n\big\|+(L+1)\|Au_{n+1}-Au_n\|+\big\|T^{n+1}Au_{n+1}-Au_{n+1}\big\|\\ &\le L\big\|Au_n-T^nAu_n\big\|+(L+1)\|A\|\|u_{n+1}-u_n\|+\big\|T^{n+1}Au_{n+1}-Au_{n+1}\big\|.\end{aligned}$$

From (3.8) and (3.11), we obtain

$$\big\|(T-I)Au_n\big\|\to 0,\quad n\to\infty.\tag{3.12}$$

Since $\{x_n\}$ is bounded, there exists $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j}\rightharpoonup z\in H_1$. Using the fact that $x_{n_j}\rightharpoonup z\in H_1$ and $\|y_n-x_n\|\to 0$, $n\to\infty$, we have that $y_{n_j}\rightharpoonup z\in H_1$. Similarly, $u_{n_j}\rightharpoonup z\in H_1$ since $\|u_n-x_n\|\to 0$, $n\to\infty$.

We next show that $z\in\bigcap_{t\ge 0}F(S(t))=C$. Assume the contrary that $z\neq S(t)z$ for some $t\ge 0$. Then, by Opial's condition, we obtain from (3.10) that

$$\begin{aligned}\liminf_{j\to\infty}\|y_{n_j}-z\| &< \liminf_{j\to\infty}\big\|y_{n_j}-S(t)z\big\|\\ &\le \liminf_{j\to\infty}\big(\big\|y_{n_j}-S(t)y_{n_j}\big\|+\big\|S(t)y_{n_j}-S(t)z\big\|\big)\\ &\le \|y_{n_j}-z\|.\end{aligned}$$

This is a contradiction. Hence, $z\in\bigcap_{t\ge 0}F(S(t))=C$. On the other hand, since A is a linear bounded operator, it follows from $u_{n_j}\rightharpoonup z\in H_1$ that $Au_{n_j}\rightharpoonup Az\in H_2$. Hence, from (3.12), we have that

$$\|TAu_{n_j}-Au_{n_j}\|=\|TAu_{n_j}-Au_{n_j}\|\to 0,\quad j\to\infty.$$

Since T is demiclosed at zero, we have that $Az\in F(T)=Q$. Hence $z\in\Omega$.

Next, we prove that $\{x_n\}$ converges strongly to z. From (3.6) and Lemma 2.1(ii), we have

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\le \|y_n-z\|^2 \\
&\le \|u_n-z\|^2 - \gamma\bigl(1-k-\gamma\|A\|^2\bigr)\bigl\|(T^n-I)Au_n\bigr\|^2 \\
&\quad + \mu_n\gamma\bigl(M+M_0\|Au_n-Az\|^2\bigr) + \gamma\xi_n \\
&\le \|u_n-z\|^2 + \mu_n\gamma\bigl(M+M_0\|Au_n-Az\|^2\bigr) + \gamma\xi_n \\
&\le \|u_n-z\|^2 + M^*\mu_n + \gamma\xi_n \\
&= \bigl\|(1-\alpha_n)(x_n-z)-\alpha_n z\bigr\|^2 + M^*\mu_n + \gamma\xi_n \\
&\le (1-\alpha_n)^2\|x_n-z\|^2 - 2\alpha_n\langle u_n-z,z\rangle + M^*\mu_n + \gamma\xi_n \\
&\le (1-\alpha_n)\|x_n-z\|^2 - 2\alpha_n\langle u_n-z,z\rangle + M^*\mu_n + \gamma\xi_n,
\end{aligned} \tag{3.13}$$

where $M^* > \gamma\sup_{n\ge1}(M+M_0\|Au_n-Az\|^2) > 0$. It is clear that $-2\langle u_n-z,z\rangle \to 0$, $n\to\infty$ and $\sum_{n=1}^{\infty} M^*\mu_n < \infty$; $\sum_{n=1}^{\infty}\gamma\xi_n < \infty$. Now, using Lemma 2.4 in (3.13), we have $\|x_n-z\|\to 0$. So $x_n\to z$ as $n\to\infty$.

Case 2. Assume that $\{\|x_n-x^*\|\}$ is not a monotonically decreasing sequence. Set $\Gamma_n = \|x_n-x^*\|^2$ and let $\tau:\mathbb{N}\to\mathbb{N}$ be a mapping for all $n\ge n_0$ (for some n_0 large enough) by

$$\tau(n) := \max\{k\in\mathbb{N}: k\ge n, \Gamma_k\le\Gamma_{k+1}\}.$$

Clearly, τ is a nondecreasing sequence such that $\tau(n)\to\infty$ as $n\to\infty$ and

$$\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}, \quad \forall n\ge n_0.$$

From (3.9), it is easy to see that

$$\lim_{n\to\infty}\bigl\|y_{\tau(n)} - S(t_{\tau(n)})y_{\tau(n)}\bigr\| = 0.$$

Furthermore, we can show that

$$\bigl\|(T-I)Au_{\tau(n)}\bigr\| \to 0, \quad n\to\infty.$$

By a similar argument as above in Case 1, we conclude immediately that $x_{\tau(n)}$, $y_{\tau(n)}$ and $u_{\tau(n)}$ weakly converge to z as $\tau(n)\to\infty$. At the same time, from (3.13), we note that for all $n\ge n_0$,

$$\begin{aligned}
0 &\le \|x_{\tau(n)+1}-z\|^2 - \|x_{\tau(n)}-z\|^2 \\
&\le \alpha_{\tau(n)}\bigl[-2\langle u_{\tau(n)}-z,z\rangle - \|x_{\tau(n)}-z\|^2\bigr] + M^*\mu_{\tau(n)} + \gamma\xi_{\tau(n)},
\end{aligned}$$

which gives

$$\|x_{\tau(n)}-z\|^2 \le -2\langle u_{\tau(n)}-z,z\rangle + M^*\mu_{\tau(n)} + \gamma\xi_{\tau(n)}.$$

Hence, we deduce that

$$\lim_{n\to\infty}\|x_{\tau(n)}-z\| = 0.$$

Therefore,

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0.$$

Furthermore, for $n \geq n_0$, it is easy to see that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is $\tau(n) < n$) because $\Gamma_j \geq \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. As a consequence, we obtain for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

This shows that $\lim \Gamma_n = 0$ and hence $\{x_n\}$ converges strongly to z. This completes the proof. □

Based on Lemma 1.2 and Example 1.3, we can deduce the following corollary from Theorem 3.1.

Corollary 3.2 *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator and $A^* : H_2 \to H_1$ be the adjoint of A. Let $\Im := \{S(t) : 0 \leq t < \infty\}$ be a one-parameter nonexpansive semigroup on H_1. Let $T : H_2 \to H_2$ be a uniformly L-Lipschitzian continuous and $(k, \{\mu_n\}, \{\xi_n\}, \phi)$-total asymptotically strict pseudocontractive mapping satisfying the following conditions*:

(i) $\sum_{n=1}^{\infty} \mu_n < \infty$; $\sum_{n=1}^{\infty} \xi_n < \infty$;

(ii) $\{\alpha_n\}$ *is a real sequence in* $(0,1)$ *such that* $\mu_n = o(\alpha_n)$, $\xi_n = o(\alpha_n)$, $\lim_{n\to\infty} \alpha_n = 0$; $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) *there exist constants* $M_0 > 0$, $M_1 > 0$ *such that* $\phi(\lambda) \leq M_0\lambda^2$, $\forall \lambda > M_1$.

Let $C := \bigcap_{t\geq 0} F(S(t)) \neq \emptyset$, $Q := F(T) \neq \emptyset$ *and* $\Omega \neq \emptyset$. *Let* $\{x_n\}$ *be the sequence generated by* $x_1 \in H_1$,

$$\begin{cases} u_n = (1 - \alpha_n)x_n, \\ y_n = u_n + \gamma A^*(T^n - I)Au_n, \\ x_{n+1} = \beta_n y_n + (1 - \beta_n)(\frac{1}{t_n}\int_0^{t_n} T(u)\,du)y_n, \quad n \geq 1, \end{cases} \tag{3.14}$$

where $\{\beta_n\} \subset (0,1)$ *and* $\gamma > 0$ *satisfy the following conditions*:

(a) $0 < \epsilon \leq \beta_n \leq b < 1$;

(b) $\gamma \in (0, \frac{1-k}{\|A\|^2})$.

If Ω *is nonempty, then the sequence* $\{x_n\}_{n=1}^{\infty}$ *converges strongly to an element of* Ω.

A strong mean convergence theorem for nonexpansive mappings was first established for odd mappings by Baillon [32] and it was later generalized to that of nonlinear semigroups by Reich [33]. It follows from the above proof that Theorem 3.1 is valid for nonexpansive mappings. Thus, we also have the following mean ergodic theorem for nonexpansive mappings in Hilbert spaces.

Corollary 3.3 *Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a bounded linear operator and $A^* : H_2 \to H_1$ be the adjoint of A. Let S be a nonexpansive mapping on H_1. Let $T : H_2 \to H_2$ be a uniformly L-Lipschitzian continuous and $(k, \{\mu_n\}, \{\xi_n\}, \phi)$-total asymptotically strict pseudocontractive mapping satisfying the following conditions*:

(i) $\sum_{n=1}^{\infty} \mu_n < \infty$; $\sum_{n=1}^{\infty} \xi_n < \infty$;

(ii) $\{\alpha_n\}$ *is a real sequence in* $(0,1)$ *such that* $\mu_n = o(\alpha_n)$, $\xi_n = o(\alpha_n)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(iii) *there exist constants* $M_0 > 0$, $M_1 > 0$ *such that* $\phi(\lambda) \leq M_0\lambda^2$, $\forall\lambda > M_1$.

Let $C := \bigcap_{t\geq 0} F(S(t)) \neq \emptyset$, $Q := F(T) \neq \emptyset$ *and* $\Omega \neq \emptyset$. *Let* $\{x_n\}$ *be the sequence generated by* $x_1 \in H_1$,

$$\begin{cases} u_n = (1-\alpha_n)x_n, \\ y_n = u_n + \gamma A^*(T^n - I)Au_n, \\ x_{n+1} = \beta_n y_n + (1-\beta_n)(\frac{1}{n+1}\sum_{j=0}^{n} S^j y_n), \quad n \geq 1, \end{cases} \tag{3.15}$$

where $\{\beta_n\} \subset (0,1)$ *and* $\gamma > 0$ *satisfy the following conditions*:

(a) $0 < \epsilon \leq \beta_n \leq b < 1$;

(b) $\gamma \in (0, \frac{1-k}{\|A\|^2})$.

If Ω *is nonempty, then the sequence* $\{x_n\}_{n=1}^{\infty}$ *converges strongly to an element of* Ω.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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